# Effective Semantics for the Modal Logics K and KT via Non-deterministic Matrices

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Thank You: Daniel and Elio

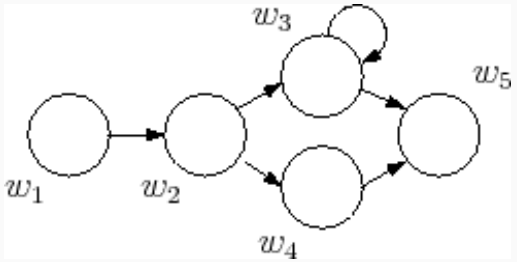
# The Modal Logic K

## Axiomatic System

- Axiom K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- Necessitation Rule: $\dfrac{\psi}{\Box\psi}$ when $\psi$ is a theorem
- extensions: T: $\Box\varphi \to \varphi$ / 4: $\Box\varphi \to \Box\Box\varphi$

## Standard Semantics – Kripke Models $(W, R, V)$

- $W$ is a set (of worlds)
- $R$ is an accessibility relation
- $v : W \to P$ is a valuation
- restrictions: reflexivity / transitivity



# Kearns Semantics [Kearns 1981]

"For I do not think there are such things as possible worlds. . ."

**4-valued Semantics for KT**

- T, t, F, f
- "not functional"

| $A$ | $\sim A$ |
|---|---|
| T | F |
| t | f |
| f | t |
| F | T |

| $A$ | $\Box A$ |
|---|---|
| T | T, t |
| t | f, F |
| f | f, F |
| F | f, F |

| $A$ | $B$ | $[A \vee B]$ |
|---|---|---|
| f | T | T |
| f | t | T, t |
| f | f | f |
| f | F | f |

Can be seen as being based on NMatrices [Avron & Lev 2005]

**Is This Enough?**

- $\Box(p \vee \neg p)$ is valid in K.
- There is a countermodel for it in this semantics:
  - $v(p) = \mathsf{f}$, $v(\neg p) = \mathsf{t}$, $v(p \vee \neg p) = \mathsf{t}$, $v(\Box(p \vee \neg p)) = \mathsf{f}$

# Kearns Semantics [Kearns 1981]

"For I do not think there are such things as possible worlds. . ."

## 4-valued Semantics for KT

- T, t, F, f
- "not functional"

| $A$ | $\sim A$ |
|---|---|
| T | F |
| t | f |
| f | t |
| F | T |

| $A$ | $\Box A$ |
|---|---|
| T | T, t |
| t | f, F |
| f | f, F |
| F | f, F |

| $A$ | $B$ | $[A \vee B]$ |
|---|---|---|
| f | T | T |
| f | t | T, t |
| f | f | f |
| f | F | f |

## Kearns' Solution: Levels

A more satisfactory definition follows.

A *0th-level T-valuation of* $L_0$ is a function which assigns one of T, t, f, F to each sentence of $L_0$ in a manner consistent with the matrices.

Let $\mathscr{V}$ be an $m$th-level T-valuation of $L_0$. $\mathscr{V}$ is an $m+1$*st level T-valuation of* $L_0$ iff $\mathscr{V}$ assigns T to every sentence $A$ which is true (which has value T or t) for every $m$th-level T-valuation.

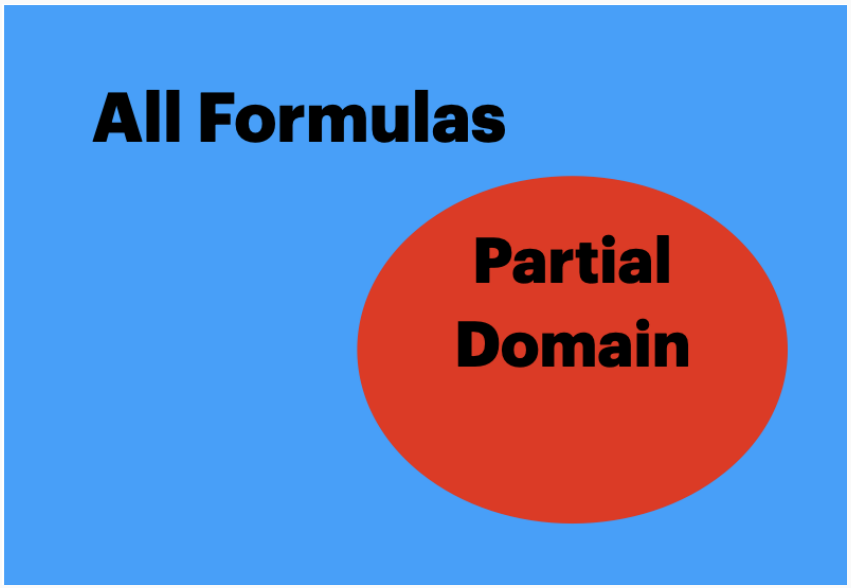
- NMatrices are just the first level
- RNMatrices [Coniglio & Toledo 2021] capture all levels
- At each level we filter out some valuations
- The semantics is the limit of this process

# After Kearns

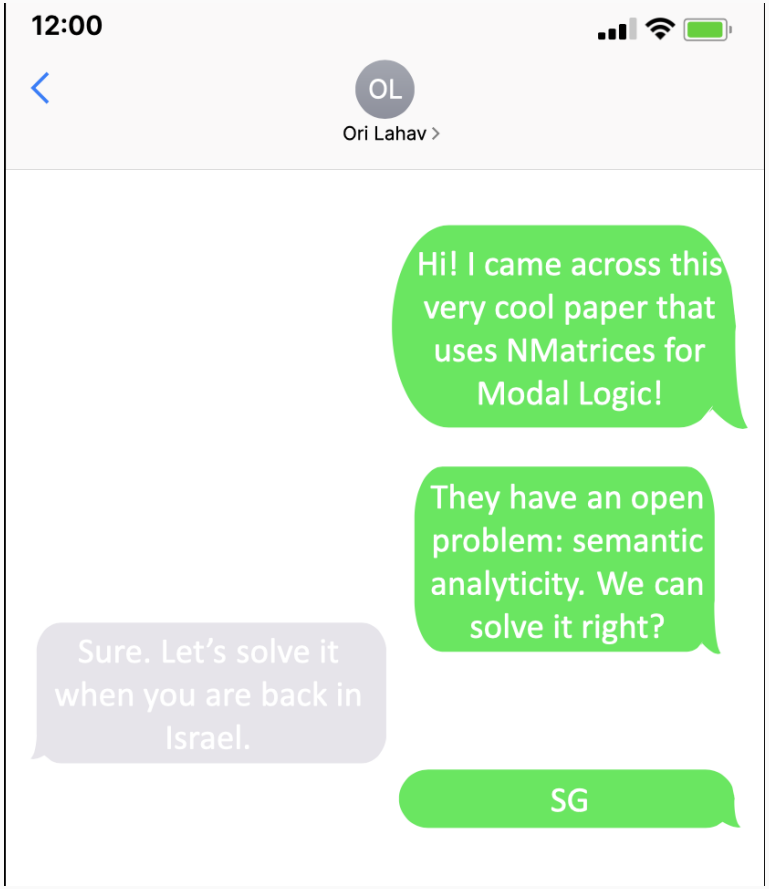
## Related Work

- Simplification, extension, NMatrices [Omori & Skurt 2016]
- More Extensions [Coniglio, del Cerro, Peron 2015]
- Missing: effectiveness / analyticity / partial valuations
- Recent: `S4` and `KT` with 3 values, effective [Grätz 2022]

| $\Box$ | $(p$ | $\rightarrow$ | $q)$ | $\rightarrow$ | $\Box$ | $(\Box$ | $p$ | $\rightarrow$ | $\Box$ | $q)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **T** | **t** | **T** | **t** | **F** | **F** | **f** | **t** | **t** | **f** | **t** |



# The Truth







## More Modal Semantics Without Possible Worlds

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# The Truth

- Semantic analyticity: Extend partial models to total ones

**Lemma 5.13.** *Let $X \subseteq \mathcal{L}$ be closed under $\prec$. Every symmetric* PKB*-strengthened partial $X$-model can be extended to a symmetric model.*

**Lemma 4.19.** *Let* **G** *be a $\prec$-analytic $\prec$-directed calculus, $v$ a* **G***-legal trivaluation and $\psi$ a formula such that $\prec[\psi] \subseteq \mathrm{supp}(v)$. Then $v$ has a* **G***-legal $\mathrm{supp}(v) \cup \{\psi\}$-determination.*

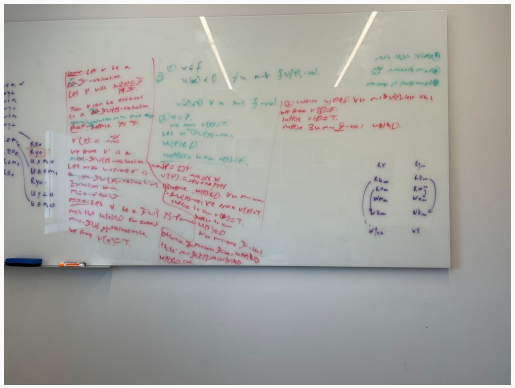
**Lemma 5.20.** *Let* **G** *be an Int-$\prec$-analytic $\prec$-directed calculus and $\mathcal{M}$ a* **G***-legal trimodel. Then there exists a* **G***-legal full determination of $\mathcal{M}$.*

THEOREM 3.9. *An $\mathcal{L}$-calculus* **G** *is $\circledcirc$-$k$-analytic iff every* **G***-legal bivaluation $v$ can be extended to a* **G***-legal $\mathcal{L}$-bivaluation, provided that $dom(v)$ is finite and closed under $\circledcirc$-$k$-subformulas.*

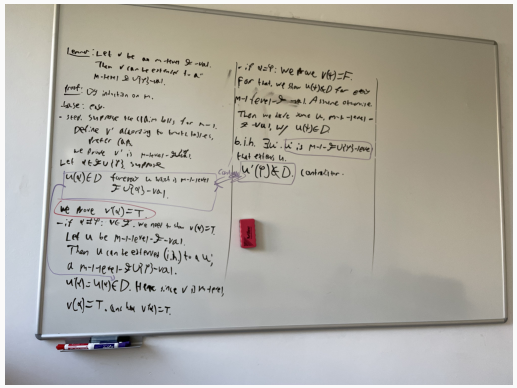
LEMMA 6.24. *Let $\mathcal{W} = \langle W, \mathcal{R}, \mathcal{V}\rangle$ be a* **G***-legal $\circledcirc$-$k$-closed biframe for $M$. Then, $\mathcal{W}$ can be extended to a* **G***-legal $\circledcirc$-$k$-closed biframe $\mathcal{W}'$ for $M$, such that $\circledcirc dom(\mathcal{W}) \subseteq dom(\mathcal{W}')$, and for every $\diamond \in \diamond^n_{\mathcal{L}} \setminus \circledcirc$, $\diamond(\varphi_1, \ldots, \varphi_n) \in dom(\mathcal{W}')$ whenever $\circledcirc^{\leq k}\{\varphi_1, \ldots, \varphi_n\} \subseteq dom(\mathcal{W})$.*



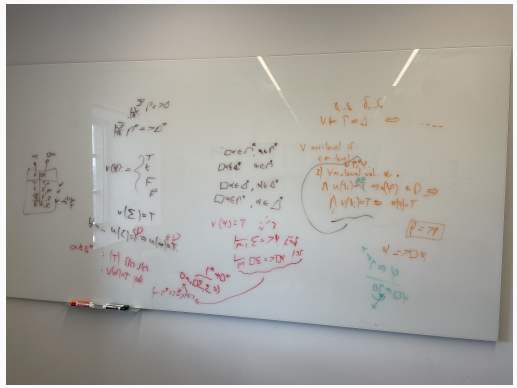
# The Truth



Extending $\infty$-level Valuations

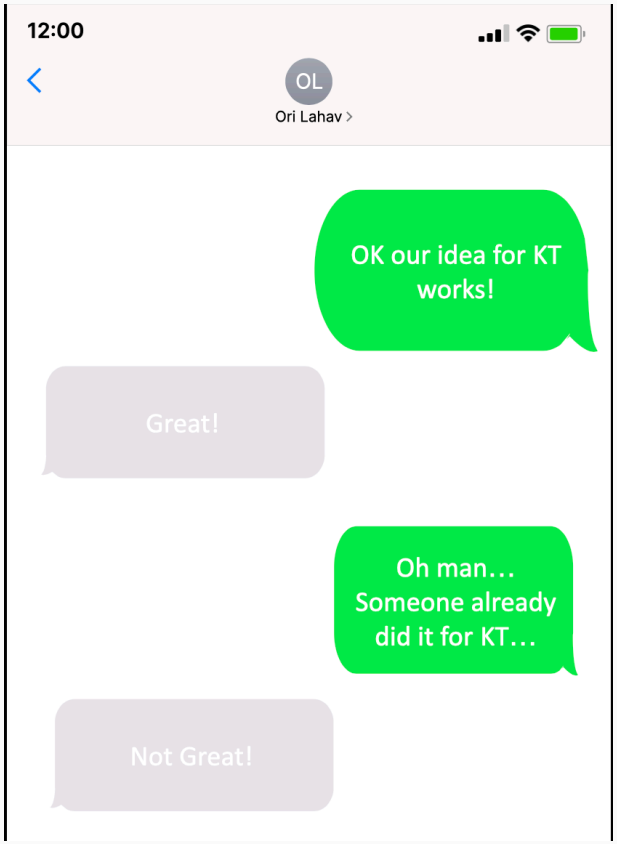


Extending $m$-level Valuations



Proof Theory Detour

# The Truth




**Truth tables for modal logics T and S4, by using three-valued non-deterministic level semantics**

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# Contributions

Still, found out that we do have some novel contributions.

- Effectiveness also for `K`
- Free Lunch: `KT`
- Natural tables
  - Like Daniel's minimal modal system
- Clean definition of partial valuations
  - No cycles
  - Declarative, not operational
- Connection to sequent calculi



# Four Values

**Intuition**

$$\mathcal{V}_4 \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}, \mathsf{f}, \mathsf{F}\} \qquad \mathcal{D} \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}\}$$

- holds in acc. worlds $\mapsto$ capital letter
- fails in acc. worlds $\mapsto$ lowecase letter
- holds here $\mapsto$ some $t$
- fails here $\mapsto$ some $f$

| | Holds in acc. worlds | Doesn't hold in acc. worlds |
|---|---|---|
| Holds here | T | t |
| Doesn't hold here | F | f |

## Truth Tables: Classical Connectives

$$\mathcal{V}_4 \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}, \mathsf{f}, \mathsf{F}\} \qquad \mathcal{D} \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}\}$$

- Expansion: $t \mapsto \{\mathsf{t}, \mathsf{T}\} \qquad f \mapsto \{\mathsf{f}, \mathsf{F}\}$

| $x \tilde{\supset} y$ | t | f |
|---|---|---|
| t | t | f |
| f | t | t |

$\mapsto$

| $x \tilde{\supset} y$ | T | t | F | f |
|---|---|---|---|---|
| T | $\mathcal{D}$ | $\mathcal{D}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| t | $\mathcal{D}$ | $\mathcal{D}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| F | $\mathcal{D}$ | $\mathcal{D}$ | $\mathcal{D}$ | $\mathcal{D}$ |
| f | $\mathcal{D}$ | $\mathcal{D}$ | $\mathcal{D}$ | $\mathcal{D}$ |

| $x \tilde{\wedge} y$ | t | f |
|---|---|---|
| t | t | f |
| f | f | f |

$\mapsto$

| $x \tilde{\wedge} y$ | T | t | F | f |
|---|---|---|---|---|
| T | $\mathcal{D}$ | $\mathcal{D}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| t | $\mathcal{D}$ | $\mathcal{D}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| F | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| f | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |

| $x \tilde{\neg} y$ | $\tilde{\neg} x$ |
|---|---|
| t | f |
| f | t |

$\mapsto$

| $x$ | $\tilde{\neg} x$ |
|---|---|
| T | $\overline{\mathcal{D}}$ |
| t | $\overline{\mathcal{D}}$ |
| F | $\mathcal{D}$ |
| f | $\mathcal{D}$ |

| $x \tilde{\vee} y$ | t | f |
|---|---|---|
| t | t | t |
| f | t | f |

$\mapsto$

| $x \tilde{\vee} y$ | T | t | F | f |
|---|---|---|---|---|
| T | $\mathcal{D}$ | $\mathcal{D}$ | $\mathcal{D}$ | $\mathcal{D}$ |
| t | $\mathcal{D}$ | $\mathcal{D}$ | $\mathcal{D}$ | $\mathcal{D}$ |
| F | $\mathcal{D}$ | $\mathcal{D}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| f | $\mathcal{D}$ | $\mathcal{D}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |

# Tables

## Truth Tables: $\Box$

$$\mathcal{V}_4 \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}, \mathsf{f}, \mathsf{F}\} \qquad \mathcal{D} \stackrel{\text{def}}{=} \{\mathsf{T}, \mathsf{t}\}$$

Intuition:

| | Holds in acc. worlds | Doesn't hold in acc. worlds |
|---|---|---|
| Holds here | T | t |
| Doesn't hold here | F | f |

$\varphi$ holds in acc. worlds iff $\Box\varphi$ holds here

$$v(\varphi) \in \{\mathsf{T}, \mathsf{F}\} \text{ iff } v(\Box\varphi) \in \mathcal{D}$$

| $x$ | $\tilde{\Box}x$ |
|---|---|
| T | $\mathcal{D}$ |
| t | $\overline{\mathcal{D}}$ |
| F | $\mathcal{D}$ |
| f | $\overline{\mathcal{D}}$ |

# Levels?

Kearns' condition for level $m + 1$:

- Every formula that is satisfied by all...
- Sequent calculi carry assumptions
- Need a stronger condition with assumptions



# Levels

| | Holds in acc. worlds | Doesn't hold in acc. worlds |
|---|---|---|
| Holds here | T | t |
| Doesn't hold here | F | f |

**A General Principle (equiv. to Lukas')**

A formula that is entailed from a set of formulas that hold in all acc. worlds should hold in all acc. worlds



$$(*)\ v^{-1}[\{\mathsf{T},\mathsf{F}\}] \vdash \varphi \implies v(\varphi) \in \{\mathsf{T},\mathsf{F}\}$$

**NMatrices Are Not Enough**

$v(p) = \mathsf{T}, v(q) = \mathsf{T}, v(p \wedge q) = \mathsf{t}$ is legal.

| $x\tilde{\wedge}y$ | T | t | F | f |
|---|---|---|---|---|
| T | $\mathcal{D}$ | $\mathcal{D}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| t | $\mathcal{D}$ | $\mathcal{D}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| F | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |
| f | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ | $\overline{\mathcal{D}}$ |

# Levels

## A General Principle

A formula that is entailed from a set of formulas that hold in all acc. worlds should hold in all acc. worlds



$$(*)\ v^{-1}[\{\mathsf{T},\mathsf{F}\}] \vdash \varphi \implies v(\varphi) \in \{\mathsf{T},\mathsf{F}\}$$

## Levels To The Rescue

$\mathbb{V}^0_{\mathrm{K}} \stackrel{\text{def}}{=} \{v \mid v \text{ respects } \mathsf{M}_{\mathrm{K}}\}$

$\mathbb{V}^{m+1}_{\mathrm{K}} \stackrel{\text{def}}{=} \{v \in \mathbb{V}^m_{\mathrm{K}} \mid \forall \varphi.\, v^{-1}[\{\mathsf{T},\mathsf{F}\}] \vdash^{\mathbb{V}^m_{\mathrm{K}}} \varphi \implies v(\varphi) \in \{\mathsf{T},\mathsf{F}\}\}$

$\mathbb{V}_{\mathrm{K}} \stackrel{\text{def}}{=} \bigcap_{m\in\mathbb{N}} \mathbb{V}^m_{\mathrm{K}}$ is a maximal set satisfying $(*)$

$\mathbb{V}_{\mathrm{K}}$ is our semantics!

# Levels

## A General Principle

A formula that is entailed from a set of formulas that hold in all acc. worlds should hold in all acc. worlds



$$(*)\ v^{-1}[\{\mathsf{T},\mathsf{F}\}] \vdash \varphi \implies v(\varphi) \in \{\mathsf{T},\mathsf{F}\}$$

## Levels To The Rescue (Partial Valuations)

$\mathbb{V}_{\mathrm{K}}^{\mathcal{F},0} \stackrel{\text{def}}{=} \{v \mid v \text{ respects } \mathsf{M}_{\mathrm{K}} \text{ with domain } \mathcal{F}\}$

$\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m+1} \stackrel{\text{def}}{=} \{v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F},m} \mid \forall \varphi \in \mathcal{F}.\, v^{-1}[\{\mathsf{T},\mathsf{F}\}] \vdash^{\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}} \varphi \implies v(\varphi) \in \{\mathsf{T},\mathsf{F}\}\}$

$\mathbb{V}_{\mathrm{K}}^{\mathcal{F}} \stackrel{\text{def}}{=} \bigcap_{m\in\mathbb{N}} \mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}$ is a maximal set satisfying $(*)$

$\mathbb{V}_{\mathrm{K}}^{\mathcal{F}}$ is our semantics!

# Main Results

## Satisfaction And Consequence

- $v \models \varphi$ if $v(\varphi) \in \mathcal{D}$
- $\mathcal{T} \vdash^{\mathbb{V}_{\mathtt{K}}} \varphi$: $\forall\ v \in \mathbb{V}_{\mathtt{K}}$, if $v \models \mathcal{T}$ then $v \models \varphi$
- Similarly: $\vdash^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}}}$ and $\vdash^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F},m}}$.

**Thm. (Soundness and Completeness)**
$\varphi$ follows from $\mathcal{T}$ in $\mathtt{K}$ iff $\mathcal{T} \vdash^{\mathbb{V}_{\mathtt{K}}} \varphi$

**Thm. (Effectiveness)**
For each $\varphi$ there is a computable $m_\varphi \in \mathbb{N}$ such that:

$$\varphi \text{ is } \mathbb{V}_{\mathtt{K}}\text{-SAT iff it is } \mathbb{V}_{\mathtt{K}}^{sub(\varphi),m_\varphi}\text{-SAT}$$





# First Result: Soundness and Completeness

**Thm. (Soundness and Completeness)**
$\varphi$ follows from $\mathcal{T}$ in $\mathtt{K}$ iff $\mathcal{T} \vdash^{\mathbb{V}_{\mathtt{K}}} \varphi$

**Sequent Calculus**

- We do not translate Kripke to Kearns (future work)
- We do not work with the Hilbert Calculus
- Instead: went through a sequent calculus for $\mathtt{K}$
- Bonus: connection between levels and applications of rule $K$



## The Sequent Calculus $\mathsf{G_K}$ [Wansing 2002]

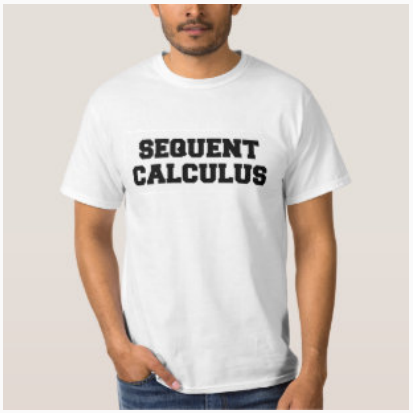
$$(\text{WEAK})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (\text{ID})\ \frac{}{\Gamma, \varphi \Rightarrow \varphi, \Delta} \qquad (\text{CUT})\ \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta} \qquad (K)\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

$$(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta} \qquad (\supset\Rightarrow)\ \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \supset \psi, \Delta} \qquad (\Rightarrow\supset)\ \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta}$$

$$(\wedge \Rightarrow)\ \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} \qquad (\vee \Rightarrow)\ \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}$$

### Properties

- Sound and complete for $\mathtt{K}$
- Subformula Property

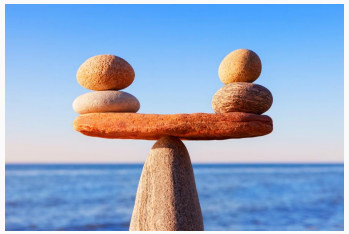


# Derivations and Levels

**Theorem:**
$\Gamma \vdash^{\mathbb{V}_K} \varphi \quad \text{iff} \vdash_{G_K} \Gamma \Rightarrow \varphi$

**Definition**

- $\vdash_{G_K} \Gamma \Rightarrow \Delta$: The sequent $\Gamma \Rightarrow \Delta$ is derivable.

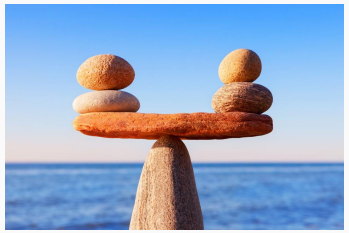


# Derivations and Levels

**Theorem:**
$\Gamma \vdash^{\mathbb{V}^m_{\mathrm{K}}} \varphi$ iff $\vdash^{m}_{\mathsf{G}_{\mathrm{K}}} \Gamma \Rightarrow \varphi$

**Definition**

- $\vdash^{m}_{\mathsf{G}_{\mathrm{K}}} \Gamma \Rightarrow \Delta$: The sequent $\Gamma \Rightarrow \Delta$ is derivable:
  - max number of $K$ applications in each branch is $m$



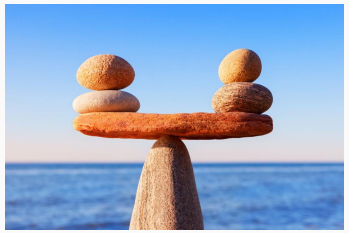
# Derivations and Levels

**Theorem:**

$\Gamma \vdash^{\mathbb{V}^{\mathcal{F},m}_{\mathrm{K}}} \varphi$ iff $\vdash^{\mathcal{F},m}_{\mathsf{G}_{\mathrm{K}}} \Gamma \Rightarrow \varphi$

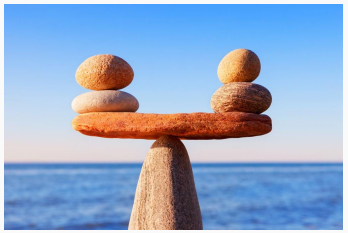
**Definition**

- $\vdash^{\mathcal{F},m}_{\mathsf{G}_{\mathrm{K}}} \Gamma \Rightarrow \Delta$: The sequent $\Gamma \Rightarrow \Delta$ is derivable:
  - max number of $K$ applications in each branch is $m$
  - using only $\mathcal{F}$-formulas



# Derivations and Levels

- $\mathbb{V}_{\mathrm{K}}$ corresponds to all derivations.
- $\mathbb{V}_{\mathrm{K}}^{m}$ corresponds to derivations with $K$-depth $m$
- $\mathbb{V}_{\mathrm{K}}^{\mathcal{F}}$ corresponds to derivations with only $\mathcal{F}$-formulas
- $\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}$ combines the two



# Soundness and Completeness Proof

## Proof

- Soundness – routine
- Completeness – relatively standard
- Here we sketch some details




## Steps

- Switch to infinite sequents (if $\mathcal{F}$ is infinite)
- Start with an underivable sequent
- Extend it to a maximal consistent (possibly infinite) sequent
- This maximal sequent induces a countermodel

# Completeness Proof

## Countermodel

$L \Rightarrow R$ is maximal consistent

For $m = 0$:

$$\lambda\varphi \in \mathcal{F}.\begin{cases} \mathsf{T} & \varphi \in L \text{ and } \Box\varphi \in L \\ \mathsf{t} & \varphi \in L \text{ and } \Box\varphi \notin L \\ \mathsf{F} & \varphi \in R \text{ and } \Box\varphi \in L \\ \mathsf{f} & \varphi \in R \text{ and } \Box\varphi \notin L \end{cases}$$

For $m > 0$:

$$\lambda\varphi \in \mathcal{F}.\begin{cases} \mathsf{T} & \varphi \in L \text{ and } \vdash^{\mathcal{F},m-1}_{\mathsf{G_K}} \mathbb{B}^L_{\mathcal{F}} \Rightarrow \varphi \\ \mathsf{t} & \varphi \in L \text{ and } \nvdash^{\mathcal{F},m-1}_{\mathsf{G_K}} \mathbb{B}^L_{\mathcal{F}} \Rightarrow \varphi \\ \mathsf{F} & \varphi \in R \text{ and } \vdash^{\mathcal{F},m-1}_{\mathsf{G_K}} \mathbb{B}^L_{\mathcal{F}} \Rightarrow \varphi \\ \mathsf{f} & \varphi \in R \text{ and } \nvdash^{\mathcal{F},m-1}_{\mathsf{G_K}} \mathbb{B}^L_{\mathcal{F}} \Rightarrow \varphi \end{cases}$$

$$\mathbb{B}^X_{\mathcal{F}} \stackrel{\text{def}}{=} \{\psi \in \mathcal{F} \mid \Box\psi \in X\}$$

- Evidently falsifies $L \Rightarrow R$
- Core of proof: countermodel is legal.
- Keeping track over $m$ and $\mathcal{F}$.



# Second Result: Effectiveness

**Thm. (Effectiveness)**
For each $\varphi$ there is a computable $m_\varphi \in \mathbb{N}$ such that:

$$\varphi \text{ is } \mathbb{V}_{\mathrm{K}}\text{-SAT iff it is } \mathbb{V}_{\mathrm{K}}^{sub(\varphi),m_\varphi}\text{-SAT}$$

- Motivation: Decision procedure based on the semantics
- Cannot iterate through all valuations
- Cannot even check if a valuation is in $\mathbb{V}_{\mathrm{K}}$ (or even in $\mathbb{V}_{\mathrm{K}}^{\mathcal{F}}$)



# Effectiveness

## Naive Algorithm

**Algorithm** Deciding $\Gamma \vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$.

```
1: for v : L → {T, t, F, f} do
2:     if v ∈ V_K and v ⊨ Γ and v ⊭ φ then
3:         return "NO"
4: return "YES"
```

**Justification**
Soundness and Completeness X2
$(\mathrm{K} \leftrightarrow \mathrm{G}_{\mathrm{K}} \leftrightarrow \mathbb{V}_{\mathrm{K}})$

**Problem**
Infinitely-many valuations to check

# Effectiveness

## Only Subformulas

**Algorithm** Deciding $\Gamma \vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$.

1: $\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
2: **for** $v : \mathcal{F} \to \{\mathsf{T}, \mathsf{t}, \mathsf{F}, \mathsf{f}\}$ **do**
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## Justification

1. $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi \Longrightarrow$ (def.)
2. $\Gamma \nvdash^{\mathbb{V}_{\mathrm{K}}^{\mathcal{F}}} \varphi \Longrightarrow$ (completeness)
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**Problem**
Infinitely-many levels to check

# Effectiveness

## Linear Bound

**Algorithm** Deciding $\Gamma \vdash^{\mathbb{V}_{\mathrm{K}}} \varphi$.

1: $\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
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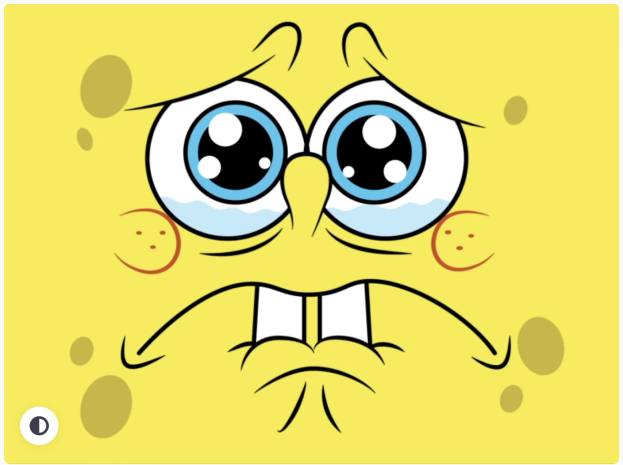
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## Problem

- Often YES/NO is not enough
- Model generation
- Is $v$ a "real" model?

# Extending Valuations vs. Replacing Valuations

**Using Soundness and Completeness – Replacing Valuations**

- Soundness and Completeness $\rightarrow$ if there is a partial model then there is a total model.
- The total model might be completely different from the partial one.
- Hence, we can say "No" but cannot return a countermodel

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**Proof**

- Do this for every $m$ separately
- Extend by a single formula:
  - Old formulas retain their value
  - The new formula is assigned according to:
    - $\mathcal{D}$ or $\overline{\mathcal{D}}$: truth tables
    - lowercase or uppercase: $v^{-1}[\mathsf{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{K}}^{\mathcal{F}\cup\{\psi\},m-1}} \psi$
- Core of the proof:
  - the extended valuation is legal
  - from $v_0, v_1, \ldots$ to a single $v$

## Exponential Bound

**Algorithm** Deciding $\Gamma \vdash^{\mathbb{V}_K} \varphi$.

1: $\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
2: $m \leftarrow 4^{|\mathcal{F}|}$
3: **for** $v : \mathcal{F} \to \{\mathsf{T}, \mathsf{t}, \mathsf{F}, \mathsf{f}\}$ **do**
4: **if** $v \in \mathbb{V}_K^{\mathcal{F},m}$ and $v \models \Gamma$ and $v \not\models \varphi$ **then**
5: **return** "NO", $v$
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## Justification

- $\mathbb{V}_K^{\mathcal{F},m} = \mathbb{V}_K^{\mathcal{F}}$
- Every $v \in \mathbb{V}_K^{\mathcal{F}}$ can be extended to some $v' \in \mathbb{V}_K$



# Trade-offs

- First algorithm:
  - No model construction
  - $m$ is the modal depth
- Second algorithm:
  - With model construction
  - $m$ is exponential

# The Modal Logic KT

| | Holds in acc. worlds | Doesn't hold in acc. worlds |
|---|---|---|
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- Everything I talked about works for the remaining three values
  - Soundness and Completeness
  - Levels represent bounded derivations
  - Replacing partial valuations
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# Ongoing Work: Other Modal Logics

- Daniel: axioms $\mapsto$ changing 0-level truth table
- Us: axioms $\mapsto$ changing the filter
  - Not for spite
  - Needed this to prove completeness
  - Nice separation: acc. relation $\leftrightarrow$ filtering condition
- Example: $K$:

$$\mathbb{V}_{\mathrm{K}}^{m+1} \stackrel{\text{def}}{=} \{v \in \mathbb{V}_{\mathrm{K}}^{m} \mid \forall \varphi.\, v^{-1}[\{\mathsf{T}, \mathsf{F}\}] \vdash^{\mathbb{V}_{\mathrm{K}}^{m}} \varphi \implies v(\varphi) \in \{\mathsf{T}, \mathsf{F}\}\}$$

- ✓ Completeness for all levels
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- ? Extending partial valuations
- Similarly for other logics ($D, B, \ldots$)

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# Conclusion

## We Have Seen

- 4-valued semantics for `K`
- Based on NMatrices (RNMatrices [Coniglio & Toledo 2021] )
- Effective, model constructing
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## Future Work

- More modal logics
- Complexity
- Implementation
- A bridge between two K's
- We are happy to collaborate :)



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Thank You!